

Primordial magnetogenesis from vector Galileons

Debottam Nandi^{1,*} and S. Shankaranarayanan¹

¹*School of Physics, Indian Institute of Science Education and Research
Thiruvananthapuram (IISER-TVM), Trivandrum 695016, India*

Like Scalar Galileons, Einstein-Hilbert action and the Lovelock extensions contain higher order derivatives in action, however their equations of motion are second order. We are lead to ask: Can there exist a corresponding action for spin-1 or electromagnetic fields? By demanding three conditions — theory be described by vector potential A^μ and its derivatives, Gauge invariance be satisfied, and equations of motion be linear in second derivatives of vector potential — we construct a higher derivative electromagnetic action which does not have ghosts and preserve gauge invariance. We show that the action breaks conformal invariance explicitly and leads to generation of magnetic field during inflation. One unique feature of our model is that appreciable magnetic fields are generated at small wavelengths while tiny magnetic fields at large wavelengths that are consistent with current observations.

I. INTRODUCTION

Since the early days of quantum electrodynamics, higher derivative field theories [1] have been proposed to improve the divergence structure. However, higher-derivative theories suffer from Ostrogradsky instability [2]. These negative energy states can be traded by negative norm states leading to non-unitary theories [3].

Recently, it has been realized that it is possible to construct scalar field theories whose action can have higher derivatives, however, the equations of motion are still second order. These are referred to as Galilean models and do not suffer from Ostrogradsky instabilities [4]. Scalar Galilean theories have a lot in common with Lovelock theories of gravity [5]. Lovelock theories are obtained by imposing three conditions — gravity must be described by metric and its derivatives, diffeomorphism invariance and equations of motion be quasi-linear. Using these conditions, it can be shown that Einstein's gravity is unique in 4-D. In higher dimensions, $R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$ also lead to quasi-linear equations of motions. Lovelock extensions of Einstein gravity are shown to be free of ghost and evade problems of Unitarity [6].

It is natural to ask: *Can we construct a higher derivative Electromagnetic (EM) field action by demanding following three conditions: theory be described by vector potential A^μ and its derivatives, Gauge invariance is satisfied and equations of motion be second order?* In this work, we explicitly construct a higher-derivative EM action satisfying the above conditions. We show that the higher-derivative terms in the action vanishes in the flat space-time and hence, do not have any observable consequences in terrestrial experiments [7]. We provide one concrete observational consequence and show that our model provides an elegant solution to the generation of primordial magnetic field during inflation.

The standard EM action in 4-D space-time,

$$\mathcal{S}_{SEM} = - \int \frac{d^4x}{4} \sqrt{-g} F_{\mu\nu} F^{\mu\nu}; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (1)$$

is conformally invariant [8]. Hence, the equations of motion of the magnetic field in FRW space-time are time independent [9]: $(\partial_\eta^2 - \nabla^2)(a^2 \mathbf{B}) = 0$. Thus, to generate magnetic field during inflation, it is necessary to break conformal invariance of the EM action. Starting from Turner and Widrow [10–12], several authors have suggested many ways to break the conformal invariance of the electromagnetic field by introducing (i) coupling of the electromagnetic field with the Ricci/Riemann vectors, (ii) non-minimal coupling of the electromagnetic field with scalar/axion/fermionic field and (iii) compactification from higher dimensional space-time. Here, we show that the higher derivative EM action generates the required seed magnetic fields at small length scales. The advantages of our model compared to earlier approaches is that the new EM action do not have any non-minimal coupling with other scalar/axion/fermionic fields, and do not require new physics. We use $(-, +, +, +)$ metric signature and natural units $\hbar = c = 1/(4\pi\epsilon_0) = 1$.

II. MODIFIED ELECTROMAGNETIC ACTION

We consider the following modification to the standard EM action (1),

$$\mathcal{S}_{VG} = \lambda \int d^4x \sqrt{-g} \epsilon^{\alpha\gamma\nu} \epsilon^{\mu\eta\beta} \nabla_{\alpha\beta} A_\gamma \nabla_{\mu\nu} A_\eta \quad (2)$$

where ∇ is covariant derivative, λ — whose dimension is inverse mass square — is the coupling constant that determines the effect of the higher-derivative terms in the propagation of the EM field. In 4-D space-time, the product of two anti-symmetric epsilon tensors is given by

$$\epsilon^{\alpha\gamma\nu} \epsilon^{\mu\eta\beta} = \begin{vmatrix} g^{\alpha\mu} & g^{\alpha\eta} & g^{\alpha\beta} \\ g^{\gamma\mu} & g^{\gamma\eta} & g^{\gamma\beta} \\ g^{\nu\mu} & g^{\nu\eta} & g^{\nu\beta} \end{vmatrix}. \quad (3)$$

* debottam, shanki@iisertvm.ac.in

Before we proceed, it is important to understand how the above action behaves in the flat Minkowski space-time: First, the contraction between the first and third indices of the epsilon tensor and the derivative of the vector potential $\epsilon^{\alpha\gamma\nu}\epsilon^{\mu\eta\beta}\nabla_{\alpha\beta}A_{\gamma}\nabla_{\mu\nu}A_{\eta}$ ensures no higher derivative terms in the equations of motion. Second, since the metric is constant, \mathcal{S}_{VG} is a boundary term and does not affect the equations of motion. This is the no-go theorem by Deffayet et al [7]. Third, the covariant derivatives ∇ are replaced by partial derivatives ∂ . Hence, in flat space-time, contraction between first two indices of the epsilon tensor and derivative of the vector potential $\epsilon^{\alpha\gamma\delta}\partial_{\alpha\delta}A_{\gamma}$ preserves gauge-invariance.

However, in curved space-time, action (2) is not a boundary term. The product of epsilon tensors (3) is a function of metric, thus the above action in curved space-time can not be written as a boundary term. Hence, the above action modifies the equations of motion of the electromagnetic field. Also, covariant derivatives of the vector potential lead to extra connection terms in the action which are not gauge-invariant. Thus, these additional terms can also lead to higher-derivative terms in the equation of motion.

The fact that the action (2) is gauge-invariant in flat space-time implies that we need to include non-minimal coupling terms of the electromagnetic potential and its derivatives with the Riemann/Ricci tensors and Ricci scalars. However, *ab initio* we do not know the non-minimal coupling terms and, we need to consider all possible terms. In Appendix A, we have listed twelve possible non-minimal coupling terms. Thus, the modification to the electromagnetic action is:

$$\mathcal{S}_{VEC} = \mathcal{S}_{VG} + \lambda \sum_{i=1}^{12} \mathcal{S}_i. \quad (4)$$

where \mathcal{S}_i 's for $i = 1, \dots, 12$ are given in Appendix A, D_i 's are the twelve unknown coefficients and are dimensionless and λ is the coupling constant in action (2).

Demanding that the above action is gauge-invariant in curved space-time and that the equations of motion do not contain higher order terms, the coefficients D_i 's can be fixed uniquely in-terms of D_1 (See [13] Appendix B). Thus, the model has one unknown coupling parameter λ that can be fixed from observations. The modified electromagnetic action is

$$\mathcal{S}_{EM} = \int d^4x \sqrt{-g} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_{\mu} J^{\mu} \right) + \mathcal{S}_{VEC} \quad (5)$$

where the first two terms correspond to the standard electromagnetic action.

This is the first key result of this work, regarding which we would like to stress the following points. First, in the flat space-time, \mathcal{S}_{VEC} becomes zero, thus the above reduces to standard electromagnetic action. Second, from the equation of motion of A_0 in the FRW background,

$$ds^2 = -N^2 d\eta^2 + a^2 d\mathbf{x}^2 \quad (6)$$

where $N(\eta)$ is an arbitrary Lapse function, the scalar potential is given by:

$$\Phi \equiv -A_0 = \frac{1}{4\pi(1-4DH^2)} \frac{\rho(\vec{r}_0)}{r} \quad (7)$$

where $D \equiv \lambda(1+3D_1)$. Thus, the effect of action (4) is to change the permittivity to $\epsilon \equiv (1-4DH^2)$ where H is the Hubble constant. The electrostatic potential still goes as inverse of the distance. Permittivity being positive provides a condition on the value of D . If D is negative all values are allowed, however, if D is positive, $4DH^2 < 1$. Thus, the modified action do not have any observable consequence in the terrestrial experiments. However, as we will show in the rest of this work, the above modified action has important consequence in the early Universe.

III. BREAKING OF CONFORMAL INVARIANCE AND CONSEQUENCES

Having discussed the model and the effect on the Coulomb potential, let us now look at the effects in the early Universe. Since the FRW background is conformally flat, the background gravitational field does not produce particles in the case of standard electromagnetic action (1) [8]. However, the modified action (4) explicitly breaks conformal invariance. The modified action can have significant contribution in the early Universe, thus leading to production of magnetic fields. Since we are interested in particle production, we consider only (4) and drop the standard EM action.

Since action (4) is gauge-invariant, we choose Coulomb gauge ($A_0 = 0$) and obtain all the physical quantities. In the FRW background (6), action (4) becomes:

$$\mathcal{S}_{VEC} = D \int d^4x \left[-2 \frac{a'^2}{N^3 a} A_i'^2 + 2 \frac{a''}{N a^2} (\partial_i A_j)^2 - 2 \frac{a' N'}{N^2 a^2} (\partial_i A_j)^2 \right]. \quad (8)$$

Varying the above action with respect to A_i and setting $N(\eta) = a(\eta)$, leads to the following equations of motion:

$$A_i'' + 2 \frac{J'}{J} A_i' - \frac{(aJ)'}{(aJ)^2} \nabla^2 A_i = 0, \quad \text{where } J \equiv \frac{\mathcal{H}}{a}. \quad (9)$$

Fourier decomposing the vector potential A_i [9], we get

$$\hat{A}_i(\eta, \mathbf{x}) = \sqrt{4\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_{\Lambda=1}^2 \epsilon_{\Lambda i}(\mathbf{k}) \left[\hat{b}_{\mathbf{k}}^{\Lambda} A_k(\eta) e^{i\mathbf{k} \cdot \mathbf{x}} + \hat{b}_{\mathbf{k}}^{\Lambda\dagger} A_k^*(\eta) e^{-i\mathbf{k} \cdot \mathbf{x}} \right], \quad (10)$$

where Λ corresponds to two orthonormal transverse polarizations and $\epsilon_{\Lambda i}$ are the polarization vectors. Substituting (10) in (9), we get

$$A_k'' + 2 \frac{J'}{J} A_k' + k^2 \frac{(aJ)'}{(aJ)^2} A_k = 0. \quad (11)$$

By fixing the initial state of the electromagnetic field, we can evaluate the vector potential at a later times.

To compare with the observations, we need to evaluate the energy density [12]. 0-0 component of the energy momentum tensor $T_{\mu\nu}$ in the FRW background (6) is

$$T_{00} = -\frac{N^2}{a^3} \frac{\delta\mathcal{L}}{\delta N}.$$

The energy density in conformal coordinates is:

$$\begin{aligned} \rho \equiv -T_0^0 = & -6D \frac{\mathcal{H}^2}{a^6} \delta^{ij} A'_i A'_j \\ & + 4D \frac{\mathcal{H}^2}{a^6} \delta^{ik} \delta^{jl} \partial_i A_j \partial_k A_l + 4D \frac{\mathcal{H}}{a^6} \delta^{ij} A'_i \nabla^2 A_j \end{aligned} \quad (12)$$

The first term is the energy density of the Electric field (ρ_E). Second and third terms are the energy densities of the magnetic field (ρ_B) and ($\rho_{B,B'}$), respectively:

Using the decomposition (10), the electric, magnetic part of the perturbation spectrum per logarithmic interval can be written as:

$$\mathcal{P}_B(k) \equiv \frac{d}{d\ln k} \langle 0 | \hat{\rho}_{B^2} | 0 \rangle = \frac{16D\mathcal{H}^2 k^5}{\pi a^6} |A_k|^2 \quad (13)$$

$$\mathcal{P}_E(k) \equiv \frac{d}{d\ln k} \langle 0 | \hat{\rho}_{E^2} | 0 \rangle = -\frac{24D\mathcal{H}^2 k^3}{\pi a^6} |A'_k|^2 \quad (14)$$

$$\mathcal{P}_{B,B'}(k) \equiv \frac{d}{d\ln k} \langle 0 | \hat{\rho}_{B,B'} | 0 \rangle = -\frac{16D\mathcal{H} k^5}{\pi a^6} A'_k A_k^* \quad (15)$$

It is important to note the following: In the standard electromagnetic action, the energy density is always positive and can be written as $(B_i B^i + E_i E^i)$. However, in our case, it is given by $D(H^2 B_i B^i - H^2 E_i E^i - H B'_i B_i)$. During most part of the evolution of the Universe, electrical conductivity is high [14], hence, electric fields decay and do not contribute to the energy density. This implies that $D > 0$.

Until now the analysis has been general and can be applied at any stage of the Universe evolution. In the

rest of this work, we calculate the energy density of the electromagnetic field during inflation. We assume that the inflation is driven a scalar field and that the energy density of the electromagnetic do not contribute to the accelerated expansion during inflation. In other words, we treat the electromagnetic field as a test field and obtain the power spectrum.

Let us first consider power-law inflation i. e. $a(t) = a_0 t^p$; $a(\eta) = a_0 (-\eta)^{1+\beta}$, where $p > 1$; $\beta \leq -2$. Note that $\beta = -2$ corresponds to de Sitter. Substituting $a(\eta)$ in (11), we have:

$$\mathcal{A}_k'' + \left(c_s^2 k^2 - \frac{(2+\beta)(3+\beta)}{\eta^2} \right) \mathcal{A}_k = 0 \quad (16)$$

where $\mathcal{A}_k \equiv J(\eta) A_k$ and $c_s \equiv -\frac{1}{1+\beta} > 0$. This is the second key result of this work. The electromagnetic perturbations do not propagate at the speed of light. This is not unusual, as the scalar perturbations in Galileon inflation also propagate less than the speed of light [15], however, the two speeds are not the same.

During power-law c_s is a constant and the solution to the above differential equation is given by:

$$\mathcal{A}_k = \sqrt{-\eta} \left[C_1 J_{\beta+\frac{5}{2}}(-c_s k \eta) + C_2 J_{-\beta-\frac{5}{2}}(-c_s k \eta) \right] \quad (17)$$

Imposing the initial condition in the sub-Hubble scales ($-k\eta \rightarrow \infty$) that the field is in vacuum state corresponds to $\mathcal{A}_k \rightarrow \frac{1}{\sqrt{2c_s k}} e^{-ic_s k \eta}$. This leads to:

$$C_1 = \sqrt{\frac{\pi}{4}} \frac{e^{i(\beta+1)\frac{\pi}{2}}}{\cos(\beta\pi)}, \quad C_2 = \sqrt{\frac{\pi}{4}} \frac{e^{-i\beta\frac{\pi}{2}}}{\cos(\beta\pi)}. \quad (18)$$

It is important to note that for $\beta \leq -5/2$, $J_{\beta+5/2}$ dominates, however, $J_{-\beta+5/2}$ dominates for $\beta \geq -5/2$.

From (17), we can obtain the spectra of the energy-densities (13, 14) and (15) at the crossing of the sound horizon ($c_s k_* = a_* H_* = \frac{1+\beta}{\eta_*}$). The magnetic part of the energy density is (see Appendix C):

$$\begin{aligned} \mathcal{P}_B = & \frac{16D}{\pi c_s^{11+2\beta}} \mathcal{F}_1(\beta) H_*^4 \left(\frac{k}{k_*} \right)^{10+2\beta} \quad \text{for } \beta < -\frac{5}{2}, \quad \mathcal{F}_1(\beta) = \frac{|C_1|^2}{2^{2\beta+5} (\Gamma(\beta+7/2))^2} \\ = & \frac{16D}{\pi c_s^{1-2\beta}} \mathcal{F}_2(\beta) H_*^4 \left(\frac{k}{k_*} \right)^{-2\beta} \quad \text{for } \beta > -\frac{5}{2}, \quad \mathcal{F}_2(\beta) = \frac{|C_2|^2}{2^{-2\beta-5} (\Gamma(-\beta-3/2))^2} \end{aligned} \quad (19)$$

This is the third key result regarding which we would like to stress the following points: First, for $\beta = -5$, the magnetic spectra is scale invariant. However, for $\beta = -5$, the electric field energy density diverges. Hence, $\beta = -5$ is ruled out as that will lead to negative energy density (since $D > 0$). Second, for $\beta \simeq -2$, the spectra is highly blue-titled. To go about understanding the consequence

of the same, the energy spectra during the slow-roll inflation [16] is given by (see Appendix D):

$$\mathcal{P}_B = \frac{8D}{\pi c_s^5} H_*^4 \left(\frac{k}{k_*} \right)^4; \quad c_s = 1 - \epsilon_1 \quad (20)$$

where ϵ_1 is the first slow-roll parameter [16]. It is interesting to note that at the start of inflation $\epsilon_1 \ll 1$

and the speed of the EM perturbations is close to unity. However, during inflation, as ϵ_1 increases, the speed of perturbations decrease, hence, leading to larger value of the energy spectrum. The condition for the exit of inflation is $\epsilon_1 = 1$ and, the speed of these perturbations vanishes and the above analysis fails at $\epsilon_1 = 1$. Our model predicts a highly blue-tilted magnetic and electric spectrum [17]. Finally, it is important to note that the power-spectrum in our model has the same blue-tilt as that of the vacuum polarization power-spectrum in the standard electromagnetic action. However, the power-spectrum evaluated here is due to particle production during inflation and depends on D and c_s [8, 10]. To fix these values and compare with observations, we need to evolve magnetic fields to the current epoch.

IV. POST INFLATIONARY EVOLUTION

Reheating is expected to convert the energy in inflation field to radiation [16] and Universe for most cosmic history has been good conductor ($\sigma \ll 1$). Assuming instantaneous reheating, the equation of motion of A_i for large wavelength modes is [9]:

$$\ddot{A}_i + \frac{\sigma + H(1 - 8D\dot{H} - 4DH^2)}{1 - 4DH^2} \dot{A}_i = 0 \quad (21)$$

where $J^i = -g^{ij}\sigma\dot{A}_j$. At late times, using Eq. (7), we have $DH^2 \ll 1$. Hence, the above equation reduces to:

$$\ddot{A}_i + \sigma \dot{A}_i = 0 \quad \Rightarrow \quad A_i = C_1(\mathbf{x})t^{-\sigma t} + C_2(\mathbf{x}), \quad (22)$$

which is same as standard EM action (1). Thus, the vector potential A_i is constant in time implying that the electric field vanishes and magnetic field decays as a^{-2} . During Radiation-dominated era, $H \propto a^{-2}$, and the energy density corresponding to S_{VEC} decays as a^{-6} . However, the energy density of the standard EM action goes as a^{-4} . At late times, only EM action (1) contributes.

V. CONSTRAINTS FROM OBSERVATIONS

To compare whether the generated magnetic field (20) has the right magnitude needed to seed galactic fields, we need to compare ρ_B with radiation background energy density $\rho_\gamma \propto T^4$. This is because, the magnetic field generated during inflation evolve as $\rho_B \propto a^{-4}$ [9, 10, 16] which is same as ρ_γ . Hence, the dimensionless quantity $r \equiv \rho_B/\rho_\gamma$ remains approximately constant and provides a convenient method to constrain the primordial magnetic field [10]. From Eq. (20), we get,

$$r \sim \frac{D}{c_s} 10^{-104} \lambda_{Mpc}^{-4} \text{eV}^2. \quad (23)$$

Note that D has dimensions of inverse mass square. The field strength required to seed galactic fields with an efficient galactic dynamo translates to $r \sim 10^{-34}$ [9, 10].

For length scales of 1Mpc , this translates to $D/c_s \sim 10^{70}$. Using the fact that permittivity has to be positive, from Eq. (7), we get $D \sim 10^{-46} \text{eV}^{-2}$. Thus, near the exit of inflation, $c_s \sim 10^{-116}$.

This is the last key result of this work and we would like to stress the following points: First, at the early epoch of inflation $\epsilon_1 \ll 1$, implying that, $c_s \sim 1$. Hence, the energy density of the magnetic fields generated at the early epoch of inflation is tiny and the magnetic fields, present at decoupling and homogeneous on scales larger than the horizon at that time is much less than the current limit of $B \leq 10 \text{nG}$ [9]. Second, appreciable seed magnetic fields are generated only close to the exit of inflation. Thus, our model naturally generates appreciable magnetic field at Mpc scale as the modes that leave the horizon close to the exit of inflation re-enter early during radiation epoch and an efficient dynamo mechanism can generate the observed magnetic field. Thus, our model generates appreciable magnetic fields *only* for smaller wavelength modes. This is the key unique feature of our model compared other proposed models for magnetogenesis.

VI. CONCLUSIONS AND DISCUSSIONS

In this work, by demanding that the theory be described by vector potential A^μ and its derivatives, Gauge invariance be satisfied, and equations of motion be linear in second derivatives of vector potential, we have constructed a higher derivative electromagnetic action that does not have ghosts and preserve gauge invariance. We have shown that the higher order terms vanish in the flat space-time and hence, consistent with the no-go theorem by Deffayet et al [7].

We have shown that the model breaks conformal invariance and hence, can generate magnetic field during inflation, the modes generated propagate less than the speed of light and the speed of propagation depends on the slow-roll parameter (20). We have explicitly shown that our model generates appreciable magnetic field for small wavelength modes ($\sim \text{Mpc}$) while the model generates tiny magnetic fields for large wavelength modes. This is one of the unique feature of our model compared other models that generate magnetic field during inflation. The energy density of the magnetic field is appreciable only at the end of inflation and hence, our model does not lead to any back-reaction.

For the inflation to exit, $\epsilon_1 = 1$. Our model can generate appreciable magnetic field near the exit and, in principle, can provide a dynamical mechanism for the exit of inflation. This precise exit calculation is under investigation.

The magnetic field spectra generated in our model is blue tilted. This should be contrasted with other models where the spectra can be fine tuned [12]. Recently, Kahniashvili et al [17], have done a detailed analysis to place constraints on the primordial magnetic field from

the cosmological data including models that have blue tilt. It is interesting to investigate how the mass dispersion $\sigma(M, z)$ behaves and its effect on the structure formation. This is currently under investigation.

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Appendix A: Non-minimal coupling terms

All possible combinations of contraction between Vector fields and Riemann tensor are given below:

$$\mathcal{S}_1 = D_1 \int d^4x \sqrt{-g} g^{\mu\nu} g^{\alpha\beta} g^{\gamma\delta} R_{\mu\nu} \nabla_\alpha A_\gamma \nabla_\beta A_\delta \quad (\text{A1})$$

$$\mathcal{S}_2 = D_2 \int d^4x \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} g^{\gamma\delta} R_{\mu\nu} \nabla_\alpha A_\gamma \nabla_\beta A_\delta \quad (\text{A2})$$

$$\mathcal{S}_3 = D_3 \int d^4x \sqrt{-g} g^{\mu\nu} g^{\alpha\beta} g^{\gamma\delta} R_{\mu\nu} \nabla_\alpha A_\beta \nabla_\gamma A_\delta \quad (\text{A3})$$

$$\mathcal{S}_4 = D_4 \int d^4x \sqrt{-g} g^{\mu\nu} g^{\alpha\delta} g^{\gamma\beta} R_{\mu\nu} \nabla_\alpha A_\beta \nabla_\gamma A_\delta \quad (\text{A4})$$

$$\mathcal{S}_5 = D_5 \int d^4x \sqrt{-g} g^{\mu\gamma} g^{\alpha\beta} g^{\nu\delta} R_{\mu\nu} \nabla_\alpha A_\beta \nabla_\gamma A_\delta \quad (\text{A5})$$

$$\mathcal{S}_6 = D_6 \int d^4x \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} g^{\gamma\delta} R_{\mu\nu} \nabla_\alpha A_\beta \nabla_\gamma A_\delta \quad (\text{A6})$$

$$\mathcal{S}_7 = D_7 \int d^4x \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} g^{\gamma\zeta} g^{\delta\eta} R_{\alpha\beta\gamma\delta} \nabla_\mu A_\nu \nabla_\zeta A_\eta \quad (\text{A7})$$

$$\mathcal{S}_8 = D_8 \int d^4x \sqrt{-g} g^{\mu\alpha} g^{\eta\beta} g^{\gamma\zeta} g^{\delta\nu} R_{\alpha\beta\gamma\delta} \nabla_\mu A_\nu \nabla_\zeta A_\eta \quad (\text{A8})$$

$$\mathcal{S}_9 = D_9 \int d^4x \sqrt{-g} g^{\alpha\beta} g^{\gamma\delta} g^{\mu\nu} R_{\alpha\beta} R_{\gamma\delta} A_\mu A_\nu \quad (\text{A9})$$

$$\mathcal{S}_{11} = D_{11} \int d^4x \sqrt{-g} g^{\alpha\gamma} g^{\beta\delta} g^{\mu\nu} R_{\alpha\beta} R_{\gamma\delta} A_\mu A_\nu \quad (\text{A11})$$

$$\mathcal{S}_{10} = D_{10} \int d^4x \sqrt{-g} g^{\alpha\beta} g^{\gamma\mu} g^{\delta\nu} R_{\alpha\beta} R_{\gamma\delta} A_\mu A_\nu \quad (\text{A10})$$

$$\mathcal{S}_{12} = D_{12} \int d^4x \sqrt{-g} g^{\alpha\gamma} g^{\beta\mu} g^{\delta\nu} R_{\alpha\beta} R_{\gamma\delta} A_\mu A_\nu \quad (\text{A12})$$

Appendix B: Details of fixing the coefficients

Evaluating the equations of motion for an arbitrary metric is hard and also non-transparent. Hence, to calculate equations of motion and thus to fix the coefficients, we consider FRW background

$$ds^2 = -N^2 d\eta^2 + a^2 d\mathbf{x}^2$$

where $N(\eta)$ is the Lapse function. The action becomes

$$\begin{aligned} \mathcal{L}_{SV} = & 4\delta^{ij} A_i \partial_{00} A_j N^{(-4)} a'^2 a^{(-4)} - 2\delta^{ij} A_i A_j \delta_{ij} N^{(-4)} a'^4 a^{(-6)} - 4\delta^{ij} A_i A_j a'' N^{(-4)} a'^2 a^{(-5)} - \\ & 8\delta^{ij} A_i \partial_0 A_j N' N^{(-5)} a'^2 a^{(-4)} + 4\delta^{ij} A_i A_j N' N^{(-5)} a'^3 a^{(-5)} - 4\delta^{ij} \partial_0 A_i a' \partial_{00} A_j N^{(-4)} a^{(-3)} + \\ & 4\delta^{ij} A_i \partial_0 A_j a' a'' N^{(-4)} a^{(-4)} + 2\delta^{ij} \partial_0 A_i \partial_0 A_j N^{(-4)} a'^2 a^{(-4)} + 4\delta^{ij} \partial_0 A_i \partial_0 A_j N' a' N^{(-5)} a^{(-3)} + \\ & 8\delta^{ij} A_i N' \partial_j A_0 N^{(-5)} a'^2 a^{(-4)} - 4\delta^{ij} A_i a' \partial_j A_0 a'' N^{(-4)} a^{(-4)} - 4\delta^{ij} \partial_0 A_i \partial_j A_0 N^{(-4)} a'^2 a^{(-4)} - \\ & 8\delta^{ij} \partial_0 A_i N' a' \partial_j A_0 N^{(-5)} a^{(-3)} + 4\delta^{ij} N' a' \partial_i A_0 \partial_j A_0 N^{(-5)} a^{(-3)} - 2\delta^{ij} \delta^{kl} \partial_{0i} A_k \partial_{0j} A_l N^{(-2)} a^{(-4)} + \\ & 4A_0 \delta^{ij} N' a' \partial_{0i} A_j N^{(-5)} a^{(-3)} - 4A_0 \delta^{ij} a'' \partial_{0i} A_j N^{(-4)} a^{(-3)} + 6\delta^{ij} \delta^{kl} a' \partial_i A_k \partial_{0j} A_l N^{(-2)} a^{(-5)} + \\ & 6A_0^2 N^{(-8)} N'^2 a'^2 a^{(-2)} - 12N' a' a'' A_0^2 N^{(-7)} a^{(-2)} + 6A_0^2 N^{(-6)} a''^2 a^{(-2)} - \\ & 6\delta^{ij} \delta^{kl} \partial_i A_k \partial_j A_l N^{(-2)} a'^2 a^{(-6)} - 4\delta^{ij} A_i \partial_{0j} A_0 N^{(-4)} a'^2 a^{(-4)} - 4\delta^{ij} a' \partial_i A_0 \partial_{0j} A_0 N^{(-4)} a^{(-3)} + \\ & 2\delta^{ij} \partial_i A_0 \partial_j A_0 N^{(-4)} a'^2 a^{(-4)} + 4\delta^{ij} \partial_0 A_i a' \partial_{0j} A_0 N^{(-4)} a^{(-3)} - \delta^{ij} \delta^{kl} \partial_{ik} A_0 \partial_{jl} A_0 N^{(-2)} a^{(-4)} - \\ & 4A_0 \delta^{ij} N' a' \partial_{ij} A_0 N^{(-5)} a^{(-3)} + 6\delta^{ij} \delta^{kl} \partial_i A_k \partial_l A_j N^{(-2)} a'^2 a^{(-6)} + \delta^{ij} \delta^{kl} \delta^{mn} \partial_{ik} A_m \partial_{jl} A_n a^{(-6)} - \\ & 2A_j \delta^{kl} \partial_{kl} A_j N^{(-2)} a'^2 a^{(-6)} + 2\delta^{ij} A_i \delta^{kl} \partial_{jk} A_l N^{(-2)} a'^2 a^{(-6)} + \delta^{ij} \delta^{kl} \partial_{0i} A_j \partial_{0k} A_l N^{(-2)} a^{(-4)} + \\ & 2\delta^{ij} \delta^{kl} \partial_{0i} A_0 \partial_{jk} A_l N^{(-2)} a^{(-4)} - 2\delta^{ij} \delta^{kl} N' \partial_i A_0 \partial_{jk} A_l N^{(-3)} a^{(-4)} - \delta^{ij} \delta^{kl} \delta^{mn} \partial_{ik} A_j \partial_{lm} A_n a^{(-6)} - \\ & 2\delta^{ij} \delta^{kl} \partial_{0i} A_j \partial_{kl} A_0 N^{(-2)} a^{(-4)} + 4A_0 \delta^{ij} a'' \partial_{ij} A_0 N^{(-4)} a^{(-3)} + 4\delta^{ij} a' \partial_i A_0 \partial_{00} A_j N^{(-4)} a^{(-3)} - \\ & 2\delta^{ij} \delta^{kl} \partial_{0i} A_0 \partial_{kl} A_j N^{(-2)} a^{(-4)} + 2\delta^{ij} \delta^{kl} N' \partial_i A_0 \partial_{kl} A_j N^{(-3)} a^{(-4)} - 2\delta^{ij} \delta^{kl} \partial_{00} A_i \partial_{jk} A_l N^{(-2)} a^{(-4)} + \\ & 2\delta^{ij} A_i \delta^{kl} a'' \partial_{jk} A_l N^{(-2)} a^{(-5)} + 2\delta^{ij} \delta^{kl} \partial_0 A_i N' \partial_{jk} A_l N^{(-3)} a^{(-4)} - 2\delta^{ij} \delta^{kl} A_i N' a' \partial_{jk} A_l N^{(-3)} a^{(-5)} + \\ & 2\delta^{ij} \delta^{kl} \delta^{mn} \partial_{ij} A_k \partial_{lm} A_n a^{(-6)} + 2\delta^{ij} \delta^{kl} \partial_{0i} A_k \partial_{jl} A_0 N^{(-2)} a^{(-4)} - 6\delta^{ij} \delta^{kl} a' \partial_i A_k \partial_{0l} A_j N^{(-2)} a^{(-5)} + \\ & \delta^{ij} \delta^{kl} \partial_{0i} A_k \partial_{0l} A_j N^{(-2)} a^{(-4)} - \delta^{ij} \delta^{kl} \delta^{mn} \partial_{ik} A_m \partial_{jn} A_l a^{(-6)} + 2\delta^{ij} \delta^{kl} \partial_{00} A_i \partial_{kl} A_j N^{(-2)} a^{(-4)} - \\ & 2\delta^{ij} A_i \delta^{kl} a'' \partial_{kl} A_j N^{(-2)} a^{(-5)} - 2\delta^{ij} \delta^{kl} \partial_0 A_i N' \partial_{kl} A_j N^{(-3)} a^{(-4)} + 2\delta^{ij} \delta^{kl} A_i N' a' \partial_{kl} A_j N^{(-3)} a^{(-5)} + \\ & \delta^{ij} \delta^{kl} \partial_{ij} A_0 \partial_{kl} A_0 N^{(-2)} a^{(-4)} - \delta^{ij} \delta^{kl} \delta^{mn} \partial_{ij} A_k \partial_{mn} A_l a^{(-6)} \end{aligned} \quad (\text{B1})$$

$$\begin{aligned}
\mathcal{L}_1 = & 6a'' N^{(-6)} \partial_0 A_0^2 a^{-1} - 6N' a' N^{(-7)} \partial_0 A_0^2 a^{-1} + 6N^{(-6)} \partial_0 A_0^2 a'^2 a^{(-2)} - \\
& 6\delta^{ij} \partial_0 A_i \partial_0 A_j a'' N^{(-4)} a^{(-3)} + 6\delta^{ij} \partial_0 A_i \partial_0 A_j N' a' N^{(-5)} a^{(-3)} - 6\delta^{ij} \partial_0 A_i \partial_0 A_j N^{(-4)} a'^2 a^{(-4)} - \\
& 6\delta^{ij} \partial_i A_0 \partial_j A_0 a'' N^{(-4)} a^{(-3)} + 6\delta^{ij} N' a' \partial_i A_0 \partial_j A_0 N^{(-5)} a^{(-3)} - 6\delta^{ij} \partial_i A_0 \partial_j A_0 N^{(-4)} a'^2 a^{(-4)} + \\
& 6\delta^{ij} \delta^{kl} \partial_i A_k \partial_j A_l a'' N^{(-2)} a^{(-5)} - 6\delta^{ij} \delta^{kl} N' a' \partial_i A_k \partial_j A_l N^{(-3)} a^{(-5)} + 6\delta^{ij} \delta^{kl} \partial_i A_k \partial_j A_l N^{(-2)} a'^2 a^{(-6)} - \\
& 12A_0 \partial_0 A_0 N' a'' N^{(-7)} a^{-1} + 12A_0 \partial_0 A_0 a' N^{(-8)} N'^2 a^{-1} - 12A_0 \partial_0 A_0 N' N^{(-7)} a'^2 a^{(-2)} + \\
& 6\delta^{ji} A_j \partial_0 A_i a' a'' N^{(-4)} a^{(-4)} - 6\delta^{ji} A_j \partial_0 A_i N' N^{(-5)} a'^2 a^{(-4)} + 6\delta^{ji} A_j \partial_0 A_i N^{(-4)} a'^3 a^{(-5)} + \\
& 6\delta^{ji} A_j a' \partial_i A_0 a'' N^{(-4)} a^{(-4)} - 6A_j \delta^{ji} N' \partial_i A_0 N^{(-5)} a'^2 a^{(-4)} + 6A_j \delta^{ji} \partial_i A_0 N^{(-4)} a'^3 a^{(-5)} - \\
& 6A_0 \delta^{ij} a' \partial_i A_j a'' N^{(-4)} a^{(-4)} + 6A_0 \delta^{ij} N' \partial_i A_j N^{(-5)} a'^2 a^{(-4)} - 6A_0 \delta^{ij} \partial_i A_j N^{(-4)} a'^3 a^{(-5)} + \\
& 6A_i \delta^{ij} \partial_0 A_j a' a'' N^{(-4)} a^{(-4)} - 6A_i \delta^{ij} \partial_0 A_j N' N^{(-5)} a'^2 a^{(-4)} + 6A_i \delta^{ij} \partial_0 A_j N^{(-4)} a'^3 a^{(-5)} + \\
& 6a'' A_0^2 N^{(-8)} N'^2 a^{-1} - 6a' A_0^2 N^{(-9)} N'^3 a^{-1} + 6A_0^2 N^{(-8)} N'^2 a'^2 a^{(-2)} - \\
& 12A_i A_j \delta^{ji} a'' N^{(-4)} a'^2 a^{(-5)} + 12A_i A_j \delta^{ji} N' N^{(-5)} a'^3 a^{(-5)} - 12A_i A_j \delta^{ji} N^{(-4)} a'^4 a^{(-6)} + \\
& 6A_i \delta^{ij} a' \partial_j A_0 a'' N^{(-4)} a^{(-4)} - 6A_i \delta^{ij} N' \partial_j A_0 N^{(-5)} a'^2 a^{(-4)} + 6A_i \delta^{ij} \partial_j A_0 N^{(-4)} a'^3 a^{(-5)} - \\
& 6A_0 \delta^{ij} a' \partial_j A_i a'' N^{(-4)} a^{(-4)} + 6A_0 \delta^{ij} N' \partial_j A_i N^{(-5)} a'^2 a^{(-4)} - 6A_0 \delta^{ij} \partial_j A_i N^{(-4)} a'^3 a^{(-5)} + \\
& 18a'' A_0^2 N^{(-6)} a'^2 a^{(-3)} - 18N' A_0^2 N^{(-7)} a'^3 a^{(-3)} + 18A_0^2 N^{(-6)} a'^4 a^{(-4)}
\end{aligned} \tag{B2}$$

$$\begin{aligned}
\mathcal{L}_2 = & 3a'' N^{(-6)} \partial_0 A_0^2 a^{-1} - 3N' a' N^{(-7)} \partial_0 A_0^2 a^{-1} - 3\delta^{ij} \partial_0 A_i \partial_0 A_j a'' N^{(-4)} a^{(-3)} + \\
& 3\delta^{ij} \partial_0 A_i \partial_0 A_j N' a' N^{(-5)} a^{(-3)} + \delta^{ij} N' a' \partial_i A_0 \partial_j A_0 N^{(-5)} a^{(-3)} - 2\delta^{ij} \partial_i A_0 \partial_j A_0 N^{(-4)} a'^2 a^{(-4)} - \\
& \delta^{ij} \partial_i A_0 \partial_j A_0 a'' N^{(-4)} a^{(-3)} - \delta^{ij} \delta^{kl} N' a' \partial_i A_l \partial_j A_k N^{(-3)} a^{(-5)} + 4\delta^{ij} \delta^{kl} \partial_i A_l \partial_j A_k N^{(-2)} a'^2 a^{(-6)} + \\
& \delta^{ij} \delta^{kl} \partial_i A_l \partial_j A_k a'' N^{(-2)} a^{(-5)} - 2\delta^{ij} \delta^{kl} \partial_i A_k \partial_j A_l N^{(-2)} a'^2 a^{(-6)} - 6A_0 \partial_0 A_0 N' a'' N^{(-7)} a^{-1} + \\
& 6A_0 \partial_0 A_0 a' N^{(-8)} N'^2 a^{-1} + 3A_j \delta^{ji} \partial_0 A_i a' a'' N^{(-4)} a^{(-4)} - 3A_j \delta^{ji} \partial_0 A_i N' N^{(-5)} a'^2 a^{(-4)} - \\
& A_j \delta^{ji} N' \partial_i A_0 N^{(-5)} a'^2 a^{(-4)} + 2A_j \delta^{ji} \partial_i A_0 N^{(-4)} a'^3 a^{(-5)} + A_j \delta^{ji} a' \partial_i A_0 a'' N^{(-4)} a^{(-4)} + \\
& A_0 \delta^{ij} N' \partial_j A_i N^{(-5)} a'^2 a^{(-4)} - 2A_0 \delta^{ij} \partial_j A_i N^{(-4)} a'^3 a^{(-5)} - A_0 \delta^{ij} a' \partial_j A_i a'' N^{(-4)} a^{(-4)} - \\
& 2A_0 \delta^{ij} \partial_i A_j N^{(-4)} a'^3 a^{(-5)} + 3A_i \delta^{ij} \partial_0 A_j a' a'' N^{(-4)} a^{(-4)} - 3A_i \delta^{ij} \partial_0 A_j N' N^{(-5)} a'^2 a^{(-4)} + \\
& 3a'' A_0^2 N^{(-8)} N'^2 a^{-1} - 3a' A_0^2 N^{(-9)} N'^3 a^{-1} - 3A_i A_j \delta^{ji} a'' N^{(-4)} a'^2 a^{(-5)} + \\
& 3A_i A_j \delta^{ji} N' N^{(-5)} a'^3 a^{(-5)} - A_i \delta^{ij} N' \partial_j A_0 N^{(-5)} a'^2 a^{(-4)} + 2A_i \delta^{ij} \partial_j A_0 N^{(-4)} a'^3 a^{(-5)} + \\
& A_i \delta^{ij} a' \partial_j A_0 a'' N^{(-4)} a^{(-4)} + A_i A_j \delta^{ij} N' N^{(-5)} a'^3 a^{(-5)} - 4A_i A_j \delta^{ij} N^{(-4)} a'^4 a^{(-6)} - \\
& A_i A_j \delta^{ij} a'' N^{(-4)} a'^2 a^{(-5)} + 2A_i A_j \delta^{ji} N^{(-4)} a'^4 a^{(-6)} + A_0 \delta^{ij} N' \partial_i A_j N^{(-5)} a'^2 a^{(-4)} - \\
& A_0 \delta^{ij} a' \partial_i A_j a'' N^{(-4)} a^{(-4)} - 3N' A_0^2 N^{(-7)} a'^3 a^{(-3)} + 6A_0^2 N^{(-6)} a'^4 a^{(-4)} + \\
& 3a'' A_0^2 N^{(-6)} a'^2 a^{(-3)}
\end{aligned} \tag{B3}$$

$$\begin{aligned}
\mathcal{L}_3 = & 6a'' N^{(-6)} \partial_0 A_0^2 a^{-1} - 6N' a' N^{(-7)} \partial_0 A_0^2 a^{-1} + 6N^{(-6)} \partial_0 A_0^2 a'^2 a^{(-2)} - \\
& 12\delta^{ij} \partial_0 A_0 \partial_i A_j a'' N^{(-4)} a^{(-3)} + 12\delta^{ij} \partial_0 A_0 N' a' \partial_i A_j N^{(-5)} a^{(-3)} - 12\delta^{ij} \partial_0 A_0 \partial_i A_j N^{(-4)} a'^2 a^{(-4)} + \\
& 6\delta^{ij} \delta^{kl} \partial_i A_j \partial_k A_l a'' N^{(-2)} a^{(-5)} - 6\delta^{ij} \delta^{kl} N' a' \partial_i A_j \partial_k A_l N^{(-3)} a^{(-5)} + 6\delta^{ij} \delta^{kl} \partial_i A_j \partial_k A_l N^{(-2)} a'^2 a^{(-6)} - \\
& 12A_0 \partial_0 A_0 N' a'' N^{(-7)} a^{-1} + 12A_0 \partial_0 A_0 a' N^{(-8)} N'^2 a^{-1} - 48A_0 \partial_0 A_0 N' N^{(-7)} a'^2 a^{(-2)} + \\
& 12A_0 \delta^{ij} N' \partial_i A_j a'' N^{(-5)} a^{(-3)} - 12A_0 \delta^{ij} a' \partial_i A_j N^{(-6)} N'^2 a^{(-3)} + 48A_0 \delta^{ij} N' \partial_i A_j N^{(-5)} a'^2 a^{(-4)} + \\
& 36A_0 \partial_0 A_0 a' a'' N^{(-6)} a^{(-2)} + 36A_0 \partial_0 A_0 N^{(-6)} a'^3 a^{(-3)} - 36A_0 \delta^{ij} a' \partial_i A_j a'' N^{(-4)} a^{(-4)} - \\
& 36A_0 \delta^{ij} \partial_i A_j N^{(-4)} a'^3 a^{(-5)} + 6a'' A_0^2 N^{(-8)} N'^2 a^{-1} - 6a' A_0^2 N^{(-9)} N'^3 a^{-1} + \\
& 42A_0^2 N^{(-8)} N'^2 a'^2 a^{(-2)} - 36N' a' a'' A_0^2 N^{(-7)} a^{(-2)} - 90N' A_0^2 N^{(-7)} a'^3 a^{(-3)} + \\
& 54a'' A_0^2 N^{(-6)} a'^2 a^{(-3)} + 54A_0^2 N^{(-6)} a'^4 a^{(-4)}
\end{aligned} \tag{B4}$$

$$\begin{aligned}
\mathcal{L}_4 = & 6a'' N^{(-6)} \partial_0 A_0^2 a^{-1} - 6N' a' N^{(-7)} \partial_0 A_0^2 a^{-1} + 6N^{(-6)} \partial_0 A_0^2 a'^2 a^{(-2)} - \\
& 12\delta^{ij} \partial_0 A_j \partial_i A_0 a'' N^{(-4)} a^{(-3)} + 12\delta^{ij} \partial_0 A_j N' a' \partial_i A_0 N^{(-5)} a^{(-3)} - 12\delta^{ij} \partial_0 A_j \partial_i A_0 N^{(-4)} a'^2 a^{(-4)} + \\
& 6\delta^{ij} \delta^{kl} \partial_i A_l \partial_k A_j a'' N^{(-2)} a^{(-5)} - 6\delta^{ij} \delta^{kl} N' a' \partial_i A_l \partial_k A_j N^{(-3)} a^{(-5)} + 6\delta^{ij} \delta^{kl} \partial_i A_l \partial_k A_j N^{(-2)} a'^2 a^{(-6)} - \\
& 12A_0 \partial_0 A_0 N' a'' N^{(-7)} a^{-1} + 12A_0 \partial_0 A_0 a' N^{(-8)} N'^2 a^{-1} - 12A_0 \partial_0 A_0 N' N^{(-7)} a'^2 a^{(-2)} + \\
& 6A_j \delta^{ji} a' \partial_i A_0 a'' N^{(-4)} a^{(-4)} - 6A_j \delta^{ji} N' \partial_i A_0 N^{(-5)} a'^2 a^{(-4)} + 6A_j \delta^{ji} \partial_i A_0 N^{(-4)} a'^3 a^{(-5)} + \\
& 6A_j \delta^{ji} \partial_0 A_i a' a'' N^{(-4)} a^{(-4)} - 6A_j \delta^{ji} \partial_0 A_i N' N^{(-5)} a'^2 a^{(-4)} + 6A_j \delta^{ji} \partial_0 A_i N^{(-4)} a'^3 a^{(-5)} - \\
& 12A_0 \delta^{ij} a' \partial_i A_j a'' N^{(-4)} a^{(-4)} + 12A_0 \delta^{ij} N' \partial_i A_j N^{(-5)} a'^2 a^{(-4)} - 12A_0 \delta^{ij} \partial_i A_j N^{(-4)} a'^3 a^{(-5)} + \\
& 6A_i \delta^{ij} a' \partial_j A_0 a'' N^{(-4)} a^{(-4)} - 6A_i \delta^{ij} N' \partial_j A_0 N^{(-5)} a'^2 a^{(-4)} + 6A_i \delta^{ij} \partial_j A_0 N^{(-4)} a'^3 a^{(-5)} + \\
& 6a'' A_0^2 N^{(-8)} N'^2 a^{-1} - 6a' A_0^2 N^{(-9)} N'^3 a^{-1} + 6A_0^2 N^{(-8)} N'^2 a'^2 a^{(-2)} - \\
& 6A_i A_j \delta^{ij} a'' N^{(-4)} a'^2 a^{(-5)} + 6A_i A_j \delta^{ij} N' N^{(-5)} a'^3 a^{(-5)} - 6A_i A_j \delta^{ij} N^{(-4)} a'^4 a^{(-6)} + \\
& 6A_i \delta^{ij} \partial_0 A_j a' a'' N^{(-4)} a^{(-4)} - 6A_i \delta^{ij} \partial_0 A_j N' N^{(-5)} a'^2 a^{(-4)} + 6A_i \delta^{ij} \partial_0 A_j N^{(-4)} a'^3 a^{(-5)} - \\
& 6A_i A_j \delta^{ji} a'' N^{(-4)} a'^2 a^{(-5)} + 6A_i A_j \delta^{ji} N' N^{(-5)} a'^3 a^{(-5)} - 6A_i A_j \delta^{ji} N^{(-4)} a'^4 a^{(-6)} + \\
& 18a'' A_0^2 N^{(-6)} a'^2 a^{(-3)} - 18N' A_0^2 N^{(-7)} a'^3 a^{(-3)} + 18A_0^2 N^{(-6)} a'^4 a^{(-4)}
\end{aligned} \tag{B5}$$

$$\begin{aligned}
\mathcal{L}_5 = & 3a'' N^{(-6)} \partial_0 A_0^2 a^{-1} - 3N' a' N^{(-7)} \partial_0 A_0^2 a^{-1} + \delta^{ij} \partial_0 A_0 N' a' \partial_j A_i N^{(-5)} a^{(-3)} - \\
& 4\delta^{ij} \partial_0 A_0 \partial_j A_i N^{(-4)} a'^2 a^{(-4)} - \delta^{ij} \partial_0 A_0 \partial_j A_i a'' N^{(-4)} a^{(-3)} + 2\delta^{ij} \partial_0 A_0 \partial_i A_j N^{(-4)} a'^2 a^{(-4)} - \\
& 3\delta^{ij} \partial_0 A_0 \partial_i A_j a'' N^{(-4)} a^{(-3)} + 3\delta^{ij} \partial_0 A_0 N' a' \partial_i A_j N^{(-5)} a^{(-3)} - \delta^{ij} \delta^{kl} N' a' \partial_j A_i \partial_k A_l N^{(-3)} a^{(-5)} + \\
& 4\delta^{ij} \delta^{kl} \partial_j A_i \partial_k A_l N^{(-2)} a'^2 a^{(-6)} + \delta^{ij} \delta^{kl} \partial_j A_i \partial_k A_l a'' N^{(-2)} a^{(-5)} - 2\delta^{ij} \delta^{kl} \partial_i A_j \partial_k A_l N^{(-2)} a'^2 a^{(-6)} - \\
& 6A_0 \partial_0 A_0 N' a'' N^{(-7)} a^{-1} + 6A_0 \partial_0 A_0 a' N^{(-8)} N'^2 a^{-1} + 3A_0 \delta^{ij} N' \partial_i A_j a'' N^{(-5)} a^{(-3)} - \\
& 3A_0 \delta^{ij} a' \partial_i A_j N^{(-6)} N'^2 a^{(-3)} - 12A_0 \partial_0 A_0 N' N^{(-7)} a'^2 a^{(-2)} + 6A_0 \partial_0 A_0 N^{(-6)} a'^3 a^{(-3)} + \\
& 12A_0 \partial_0 A_0 a' a'' N^{(-6)} a^{(-2)} + A_0 \delta^{ij} N' \partial_i A_j N^{(-5)} a'^2 a^{(-4)} - 3A_0 \delta^{ij} a' \partial_i A_j a'' N^{(-4)} a^{(-4)} - \\
& A_0 \delta^{ij} a' \partial_j A_i N^{(-6)} N'^2 a^{(-3)} + 7A_0 \delta^{ij} N' \partial_j A_i N^{(-5)} a'^2 a^{(-4)} + A_0 \delta^{ij} N' \partial_j A_i a'' N^{(-5)} a^{(-3)} + \\
& 3a'' A_0^2 N^{(-8)} N'^2 a^{-1} - 3a' A_0^2 N^{(-9)} N'^3 a^{-1} + 12A_0^2 N^{(-8)} N'^2 a'^2 a^{(-2)} - \\
& 15N' A_0^2 N^{(-7)} a'^3 a^{(-3)} - 12N' a' a'' A_0^2 N^{(-7)} a^{(-2)} - 12A_0 \delta^{ij} \partial_j A_i N^{(-4)} a'^3 a^{(-5)} - \\
& 3A_0 \delta^{ij} a' \partial_j A_i a'' N^{(-4)} a^{(-4)} + 18A_0^2 N^{(-6)} a'^4 a^{(-4)} + 9a'' A_0^2 N^{(-6)} a'^2 a^{(-3)}
\end{aligned} \tag{B6}$$

$$\begin{aligned}
\mathcal{L}_6 = & 3a'' N^{(-6)} \partial_0 A_0^2 a^{-1} - 3N' a' N^{(-7)} \partial_0 A_0^2 a^{-1} - 3\delta^{ij} \partial_0 A_j \partial_i A_0 a'' N^{(-4)} a^{(-3)} + \\
& 3\delta^{ij} \partial_0 A_j N' a' \partial_i A_0 N^{(-5)} a^{(-3)} + \delta^{ij} \partial_0 A_i N' a' \partial_j A_0 N^{(-5)} a^{(-3)} - 4\delta^{ij} \partial_0 A_i \partial_j A_0 N^{(-4)} a'^2 a^{(-4)} - \\
& \delta^{ij} \partial_0 A_i \partial_j A_0 a'' N^{(-4)} a^{(-3)} + 2\delta^{ij} \partial_0 A_j \partial_i A_0 N^{(-4)} a'^2 a^{(-4)} - \delta^{ij} \delta^{kl} N' a' \partial_j A_l \partial_k A_i N^{(-3)} a^{(-5)} + \\
& 4\delta^{ij} \delta^{kl} \partial_j A_l \partial_k A_i N^{(-2)} a'^2 a^{(-6)} + \delta^{ij} \delta^{kl} \partial_j A_l \partial_k A_i a'' N^{(-2)} a^{(-5)} - 2\delta^{ij} \delta^{kl} \partial_i A_l \partial_k A_j N^{(-2)} a'^2 a^{(-6)} - \\
& 6A_0 \partial_0 A_0 N' a'' N^{(-7)} a^{-1} + 6A_0 \partial_0 A_0 a' N^{(-8)} N'^2 a^{-1} - A_j \delta^{ji} N' \partial_i A_0 N^{(-5)} a'^2 a^{(-4)} + \\
& 2A_j \delta^{ji} \partial_i A_0 N^{(-4)} a'^3 a^{(-5)} + A_j \delta^{ji} a' \partial_i A_0 a'' N^{(-4)} a^{(-4)} + 3A_j \delta^{ji} \partial_0 A_i a' a'' N^{(-4)} a^{(-4)} - \\
& 3A_j \delta^{ji} \partial_0 A_i N' N^{(-5)} a'^2 a^{(-4)} + A_0 \delta^{ij} N' \partial_i A_j N^{(-5)} a'^2 a^{(-4)} - A_0 \delta^{ij} a' \partial_i A_j a'' N^{(-4)} a^{(-4)} + \\
& 3A_i \delta^{ij} a' \partial_j A_0 a'' N^{(-4)} a^{(-4)} - 3A_i \delta^{ij} N' \partial_j A_0 N^{(-5)} a'^2 a^{(-4)} + 3a'' A_0^2 N^{(-8)} N'^2 a^{-1} - \\
& 3a' A_0^2 N^{(-9)} N'^3 a^{-1} - 4A_i A_j \delta^{ij} a'' N^{(-4)} a'^2 a^{(-5)} + 4A_i A_j \delta^{ij} N' N^{(-5)} a'^3 a^{(-5)} - \\
& A_i \delta^{ij} \partial_0 A_j N' N^{(-5)} a'^2 a^{(-4)} + 2A_i \delta^{ij} \partial_0 A_j N^{(-4)} a'^3 a^{(-5)} + A_i \delta^{ij} \partial_0 A_j a' a'' N^{(-4)} a^{(-4)} - \\
& 4A_i A_j \delta^{ij} N^{(-4)} a'^4 a^{(-6)} + 2A_i A_j \delta^{ji} N^{(-4)} a'^4 a^{(-6)} + A_0 \delta^{ij} N' \partial_j A_i N^{(-5)} a'^2 a^{(-4)} - \\
& 4A_0 \delta^{ij} \partial_j A_i N^{(-4)} a'^3 a^{(-5)} - A_0 \delta^{ij} a' \partial_j A_i a'' N^{(-4)} a^{(-4)} - 3N' A_0^2 N^{(-7)} a'^3 a^{(-3)} + \\
& 6A_0^2 N^{(-6)} a'^4 a^{(-4)} + 3a'' A_0^2 N^{(-6)} a'^2 a^{(-3)}
\end{aligned} \tag{B7}$$

$$\begin{aligned}
\mathcal{L}_7 = & -\delta^{ij} \partial_0 A_i N' a' \partial_j A_0 N^{(-5)} a^{(-3)} + \delta^{ij} \partial_0 A_i \partial_j A_0 a'' N^{(-4)} a^{(-3)} + \delta^{ij} \partial_0 A_i \partial_0 A_j N' a' N^{(-5)} a^{(-3)} - \\
& \delta^{ij} \partial_0 A_i \partial_0 A_j a'' N^{(-4)} a^{(-3)} + \delta^{ij} N' a' \partial_i A_0 \partial_j A_0 N^{(-5)} a^{(-3)} - \delta^{ij} \partial_i A_0 \partial_j A_0 a'' N^{(-4)} a^{(-3)} - \\
& \delta^{ij} \partial_0 A_j N' a' \partial_i A_0 N^{(-5)} a^{(-3)} + \delta^{ij} \partial_0 A_j \partial_i A_0 a'' N^{(-4)} a^{(-3)} + \delta^{ij} \delta^{kl} \partial_k A_i \partial_l A_j N^{(-2)} a'^2 a^{(-6)} - \\
& \delta^{ij} \delta^{kl} \partial_j A_l \partial_k A_i N^{(-2)} a'^2 a^{(-6)}
\end{aligned} \tag{B8}$$

$$\begin{aligned}
\mathcal{L}_8 = & -\delta^{ij}\partial_0 A_0 N' a' \partial_j A_i N^{(-5)} a^{(-3)} + \delta^{ij}\partial_0 A_0 \partial_j A_i a'' N^{(-4)} a^{(-3)} + \delta^{ij}\partial_0 A_i \partial_0 A_j N' a' N^{(-5)} a^{(-3)} - \\
& \delta^{ij}\partial_0 A_i \partial_0 A_j a'' N^{(-4)} a^{(-3)} + \delta^{ij}N' a' \partial_i A_0 \partial_j A_0 N^{(-5)} a^{(-3)} - \delta^{ij}\partial_i A_0 \partial_j A_0 a'' N^{(-4)} a^{(-3)} - \\
& \delta^{ij}\partial_0 A_0 N' a' \partial_i A_j N^{(-5)} a^{(-3)} + \delta^{ij}\partial_0 A_0 \partial_i A_j a'' N^{(-4)} a^{(-3)} + \delta^{ij}\delta^{kl}\partial_k A_j \partial_l A_i N^{(-2)} a'^2 a^{(-6)} - \\
& \delta^{ij}\delta^{kl}\partial_j A_i \partial_k A_l N^{(-2)} a'^2 a^{(-6)} - 3 A_i \delta^{ij}\partial_0 A_j N' N^{(-5)} a'^2 a^{(-4)} + 2 A_i \delta^{ij}\partial_0 A_j a' a'' N^{(-4)} a^{(-4)} + \\
& A_j \delta^{ji}\partial_0 A_i N' N^{(-5)} a'^2 a^{(-4)} + A_0 \delta^{ij}a' \partial_i A_j N^{(-6)} N'^2 a^{(-3)} - A_0 \delta^{ij}N' \partial_i A_j a'' N^{(-5)} a^{(-3)} - \\
& A_j \delta^{ji}N' \partial_i A_0 N^{(-5)} a'^2 a^{(-4)} + A_j \delta^{ji}a' \partial_i A_0 a'' N^{(-4)} a^{(-4)} + 6 A_0 \partial_0 A_0 N' N^{(-7)} a'^2 a^{(-2)} - \\
& 6 A_0 \partial_0 A_0 a' a'' N^{(-6)} a^{(-2)} + 2 A_0 \delta^{ij}\partial_i A_j N^{(-4)} a'^3 a^{(-5)} + A_0 \delta^{ij}a' \partial_j A_i N^{(-6)} N'^2 a^{(-3)} - \\
& A_0 \delta^{ij}N' \partial_j A_i a'' N^{(-5)} a^{(-3)} + A_i A_j \delta^{ij}N' N^{(-5)} a'^3 a^{(-5)} - A_i A_j \delta^{ij}a'' N^{(-4)} a'^2 a^{(-5)} - \\
& 6 A_0^2 N^{(-8)} N'^2 a'^2 a^{(-2)} + 6 N' a' a'' A_0^2 N^{(-7)} a^{(-2)} - A_i \delta^{ij}N' \partial_j A_0 N^{(-5)} a'^2 a^{(-4)} + \\
& A_i \delta^{ij}a' \partial_j A_0 a'' N^{(-4)} a^{(-4)} + A_i A_j \delta^{ji}N' N^{(-5)} a'^3 a^{(-5)} - A_i A_j \delta^{ji}a'' N^{(-4)} a'^2 a^{(-5)} + \\
& 2 A_0 \delta^{ij}\partial_j A_i N^{(-4)} a'^3 a^{(-5)} - 6 A_0^2 N^{(-6)} a'^4 a^{(-4)}
\end{aligned} \tag{B9}$$

$$\begin{aligned}
\mathcal{L}_9 = & -36 A_0^2 N^{(-6)} a''^2 a^{(-2)} + 72 N' a' a'' A_0^2 N^{(-7)} a^{(-2)} - 36 A_0^2 N^{(-8)} N'^2 a'^2 a^{(-2)} + \\
& 36 A_i A_j \delta^{ij} N^{(-4)} a''^2 a^{(-4)} - 72 A_i A_j \delta^{ij} N' a' a'' N^{(-5)} a^{(-4)} + 36 A_i A_j \delta^{ij} N^{(-6)} N'^2 a'^2 a^{(-4)} - \\
& 72 a'' A_0^2 N^{(-6)} a'^2 a^{(-3)} + 72 N' A_0^2 N^{(-7)} a'^3 a^{(-3)} + 72 A_i A_j \delta^{ij} a'' N^{(-4)} a'^2 a^{(-5)} - \\
& 72 A_i A_j \delta^{ij} N' N^{(-5)} a'^3 a^{(-5)} - 36 A_0^2 N^{(-6)} a'^4 a^{(-4)} + 36 A_i A_j \delta^{ij} N^{(-4)} a'^4 a^{(-6)}
\end{aligned} \tag{B10}$$

$$\begin{aligned}
\mathcal{L}_{10} = & -18 A_0^2 N^{(-6)} a''^2 a^{(-2)} + 36 N' a' a'' A_0^2 N^{(-7)} a^{(-2)} - 18 A_0^2 N^{(-8)} N'^2 a'^2 a^{(-2)} - \\
& 12 A_i A_j \delta^{ij} N^{(-4)} a''^2 a^{(-4)} + 18 A_i A_j \delta^{ij} a'' N^{(-4)} a'^2 a^{(-5)} + 6 A_i A_j \delta^{ij} N^{(-4)} a''^2 a^{(-4)} + \\
& 6 A_i A_j \delta^{ij} N^{(-6)} N'^2 a'^2 a^{(-4)} - 18 A_i A_j \delta^{ij} N' N^{(-5)} a'^3 a^{(-5)} - 18 a'' A_0^2 N^{(-6)} a'^2 a^{(-3)} + \\
& 18 N' A_0^2 N^{(-7)} a'^3 a^{(-3)} + 12 A_i A_j \delta^{ij} N^{(-4)} a'^4 a^{(-6)}
\end{aligned} \tag{B11}$$

$$\begin{aligned}
\mathcal{L}_{11} = & -12 A_0^2 N^{(-6)} a''^2 a^{(-2)} + 24 N' a' a'' A_0^2 N^{(-7)} a^{(-2)} - 12 A_0^2 N^{(-8)} N'^2 a'^2 a^{(-2)} + \\
& 12 A_i A_j \delta^{ij} N^{(-4)} a''^2 a^{(-4)} - 24 A_i A_j \delta^{ij} N' a' a'' N^{(-5)} a^{(-4)} + 12 A_i A_j \delta^{ij} N^{(-6)} N'^2 a'^2 a^{(-4)} + \\
& 12 N' A_0^2 N^{(-7)} a'^3 a^{(-3)} - 12 A_0^2 N^{(-6)} a'^4 a^{(-4)} - 12 a'' A_0^2 N^{(-6)} a'^2 a^{(-3)} - \\
& 12 A_i A_j \delta^{ij} N' N^{(-5)} a'^3 a^{(-5)} + 12 A_i A_j \delta^{ij} N^{(-4)} a'^4 a^{(-6)} + 12 A_i A_j \delta^{ij} a'' N^{(-4)} a'^2 a^{(-5)}
\end{aligned} \tag{B12}$$

$$\begin{aligned}
\mathcal{L}_{12} = & -9 A_0^2 N^{(-6)} a''^2 a^{(-2)} + 18 N' a' a'' A_0^2 N^{(-7)} a^{(-2)} - 9 A_0^2 N^{(-8)} N'^2 a'^2 a^{(-2)} + \\
& A_i A_j \delta^{ij} N^{(-6)} N'^2 a'^2 a^{(-4)} - 4 A_i A_j \delta^{ij} N' N^{(-5)} a'^3 a^{(-5)} - 2 A_i A_j \delta^{ij} N' a' a'' N^{(-5)} a^{(-4)} + \\
& 4 A_i A_j \delta^{ij} N^{(-4)} a'^4 a^{(-6)} + 4 A_i A_j \delta^{ij} a'' N^{(-4)} a'^2 a^{(-5)} + A_i A_j \delta^{ij} N^{(-4)} a''^2 a^{(-4)}
\end{aligned} \tag{B13}$$

where the action is defined as:

$$\begin{aligned}
\mathcal{S}_i &= D_i \int d^4 x \sqrt{-g} \mathcal{L}_i \\
&= D_i \int d^4 x N a^3 \mathcal{L}_i
\end{aligned} \tag{B14}$$

Varying these along with (2) w.r.t A_μ in the FRW background, and demanding that the action is gauge-invariant and that the equation of motion only contains derivatives upto second order, the coefficients D_i are given below:

$$\begin{aligned}
D_2 &= 2, \quad D_4 = -D_1 - D_3, \quad D_6 = -2 - D_5, \quad D_{10} = -\frac{1}{6}, \\
D_7 &= -4 - 6D_1 - 6D_3 - 2D_5, \quad D_8 = 6D_3 + 2D_5, \quad D_{12} = 1 \\
D_9 &= \frac{1}{12} - \frac{D_3}{2} - \frac{D_5}{12}, \quad D_{11} = -\frac{1}{4} + \frac{3D_3}{2} + \frac{D_5}{4},
\end{aligned} \tag{B15}$$

Appendix C: Spectrum of electric and $B.B'$

Electric part and $B.B'$ part of the energy density at sound horizon become

$$\begin{aligned} \mathcal{P}_E = & -\frac{24 D}{\pi c_s^{7+2\beta}} \mathcal{G}_1(\beta) H_*^4 \left(\frac{k}{k_*}\right)^{2\beta+8}, \beta < -\frac{5}{2}, \quad \mathcal{G}_1(\beta) = \frac{|C_1|^2}{2^{2\beta+3} (\Gamma(\beta + 5/2))^2} \\ & -\frac{24 D}{\pi c_s^{1-2\beta}} \mathcal{G}_2(\beta) H_*^4 \left(\frac{k}{k_*}\right)^{2-2\beta}, \beta > -\frac{5}{2}, \quad \mathcal{G}_2(\beta) = \frac{|C_2|^2}{2^{-2\beta-3} (\Gamma(-\beta - 1/2))^2} \end{aligned} \quad (C1)$$

$$\begin{aligned} \mathcal{P}_{B.B'} = & -\frac{16 D}{\pi c_s^{10+2\beta}} \mathcal{J}_1(\beta) H_*^4 \left(\frac{k}{k_*}\right)^{2\beta+10}, \beta < -\frac{5}{2}, \quad \mathcal{J}_1(\beta) = \frac{|C_1|^2}{2^{2\beta+4} (-\beta - 5/2) (\Gamma(\beta + 5/2))^2} \\ & -\frac{16 D}{\pi c_s^{2-2\beta}} \mathcal{J}_2(\beta) H_*^4 \left(\frac{k}{k_*}\right)^{2-2\beta}, \beta > -\frac{5}{2}, \quad \mathcal{J}_2(\beta) = \frac{|C_2|^2}{2^{-2\beta-4} (-\beta - 3/2) (\Gamma(-\beta - 3/2))^2} \end{aligned} \quad (C2)$$

Appendix D: Slow-roll inflation and spectrum of the energy densities

In case of slow-roll inflation, the slow-roll parameters are defined as

$$\begin{aligned} \epsilon_1 = & -\frac{\dot{H}}{H^2}, \quad \text{where dot used for cosmic time } t \\ \epsilon_{n+1} = & \frac{\dot{\epsilon}_n}{H\epsilon}, \quad n \text{ is natural numbers.} \end{aligned} \quad (D1)$$

In conformal coordinate,

$$\epsilon_1 - 1 = -\frac{\mathcal{H}'}{\mathcal{H}^2} \quad (D2)$$

$$\Rightarrow c_s \equiv \frac{\mathcal{H}'}{\mathcal{H}^2} = 1 - \epsilon_1 \quad (\text{exact}) \quad (D3)$$

$$(D4)$$

and

$$\frac{J''}{J} = \mathcal{H}^2 (-\epsilon_1 + 2\epsilon_1^2 - \epsilon_1\epsilon_2) \quad (\text{exact}) \quad (D5)$$

In case of leading order slow-roll approximation with $\epsilon_n \ll 1$, $\mathcal{H} = -\frac{1+\epsilon_1}{\eta}$, thus

$$\frac{J''}{J} = \frac{\epsilon_1(\epsilon_2 - 1)}{\eta^2} \quad (D6)$$

Hence, the equation for \mathcal{A}_k becomes

$$\mathcal{A}_k'' + \left(c_s^2 k^2 - \frac{\epsilon_1(\epsilon_2 - 1)}{\eta^2} \right) \mathcal{A}_k = 0 \quad (D7)$$

Solution for the above equation using Bunch-Davies vacuum becomes

$$\mathcal{A}_k = \frac{\pi}{4} \sqrt{-\eta} H_\nu^1(-c_s k \eta), \quad \nu = \frac{1}{2} \sqrt{1 + \epsilon_1(\epsilon_2 - 1)} \quad (D8)$$

Using the above solution for $-k\eta \rightarrow 0$, at sound horizon, spectral energy density of the magnetic field becomes

$$\mathcal{P}_B = \frac{8 D}{\pi c_s^5} H_*^4 \left(\frac{k}{k_*}\right)^4 \quad (D9)$$